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TWO-DIMENSIONAL SOLIDIFICATION IN PIPES OF RECTANGULAR SECTION

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Abstract — A conformal transformation method is applied to the determination of the shape of the interface between a solidified layer formed on the inside of a cooled pipe of rectangular cross-section and a warmer flowing liquid. The convective heat transfer from the liquid to the interface, in the steady state case, balances the heat transfer by conduction through the solidified layer. This heat transfer, for pipes of various aspect ratio is computed as a function of a non-dimensional parameter involving the interfacial convection coefficient and the thermal conductivity of the layer. It is found that the extent of solidification displays a 'critical thickness' characteristic.

NOMENCLATURE

A, B, C, D, G, K, parameters used in the evaluation of various integrals in the Appendix;
a, dimensionless width of pipe, a'/α;
b, dimensionless height of pipe, b'/α;
d, solidified layer thickness [distance 4, 3 in Fig. 1(b)];
E_1, E_2, E_3, parameters related to integrals in text;
F, Legendre integral of the first kind;
h, interfacial heat transfer coefficient;
$I_1, I_2, I_3, I_4, I_5, I_6$, various integrals in text;
k, thermal conductivity of solidified layer;
K_1, $k(T'_f - T'_w)/h'_a(T'_l - T'_f)$;
K_2, $k(T'_f - T'_w)/h'_b(T'_l - T'_f)$;
L, length of pipe;
m, n, parameters in the t-plane;
q, heat flux;
Q, heat transfer rate through solidified layer;
$\bar{Q}$, $Q/kL(T'_f - T'_w)$;
s, coordinate normal to interface;
t, auxiliary half-plane;
T, dimensionless temperature $(T' - T'_w)/(T'_f - T'_w)$;
W, complex function $-T + i\psi$;
x, y, dimensionless coordinates, x'/α, y'/α;
z, $x + iy$.

Greek symbols

α, $k(T'_f - T'_w)/h(T'_l - T'_f)$;
ψ, coordinate in W-plane;
ω, complex function $= \mathrm{d}W/\mathrm{d}z$;
θ, $\arg \omega$;
Ω, $\log \mathrm{d}W/\mathrm{d}z$;
ϕ, amplitude angle in Legendre function.

Subscripts

l, refers to properties of the liquid;
f, refers to properties of the interface;
w, refers to properties of the pipe wall;

Primes refer to quantities in dimensional form in Fig. 1(a).

INTRODUCTION

THE SOLIDIFICATION of layers adjacent to solid surfaces occurs in many industrial situations including, for example, the icing in cryogenic installations, the freezing of liquid metals and foods, and geophysical phenomena such as lava flow. The problem discussed in this paper is concerned with the solidification of a layer inside a pipe of rectangular cross-section when the pipe wall is cooled and a warmer flowing liquid passes along the axis of the pipe. The resultant interface has a form which is unknown *a priori* and the purpose here is to locate the interface for various imposed boundary conditions, and to compute the corresponding heat transfer through the layer.

A survey of the available literature indicates that various attempts have been made to solve the so-called 'free boundary problem' but with few exceptions the problems have been tackled on a one-dimensional basis. In two-dimensions several approximate and numerical methods have been related in cases where various wall geometries were chosen. An approximate integral-method of boundary layer theory was applied by Poots [1] to determine the location and time-history of the solidification front for a square-section prism. These results were in general agreement with those of Allen and Severn [2] obtained using a relaxation method, and where the liquid phase was assumed to be at the fusion temperature. The effect of liquid solidification at the inner surface of a circular pipe on the heat transfer and on the pressure drop, was considered by Zerkle and Sunderland [3]. Experimentation showed that the effects of free convection within the flowing liquid could produce a significant departure from theory. Siegel [4] used a conformal transformation technique to determine the shape of the two-dimensional solidified layer formed on a cooled plate immersed in a warm flowing liquid. The results in the form of non-dimensional parameters provided the inspiration for this paper. A good approximation to the solidification interface was obtained by Stephan [5] by assuming a parabolic temperature distribution in the solid phase which satisfies all boundary conditions, but satisfies the heat-conduction equation only at the interface. A solution to the transient case of freezing of a liquid flowing along a circular pipe was obtained by Özisik and Mulligan [6]. In obtaining the solution certain assumptions were made which necessarily restricted the analysis to regions where the rate of change of layer thickness was small. The transient problem on a wall maintained at constant temperature was analysed by Savino and Siegel [7] using an iteration technique which displayed rapid convergence. Lazaridis [8] applied the Murray–Landis [9] method to analyse the multi-dimensional solidification problem in a square region. Comparison with existing solutions showed satisfactory agreement. Rathjen and Jiji [10] presented an analytical solution to the two-dimensional Neumann problem in a corner using the method attributed to Lightfoot [11]. This method was also applied by Budhia and Kreith [12] to freezing or melting in a wedge shaped enclosure the surfaces of which were maintained at uniform, but not necessarily equal, temperatures. Kroeger and Ostrach [13] simulated a continuous metal casting process taking into consideration the effects of natural convection in the liquid pool. It was found, for the range of parameters considered, that even for strong natural convection there was little or no effect on the position of the interface. Saitoh [14] extended the change of variable technique of Landau, to multi-dimensional problems of freezing in arbitrary domains. Examples of the Stefan type freezing were performed in regular squares, triangles and ellipses.

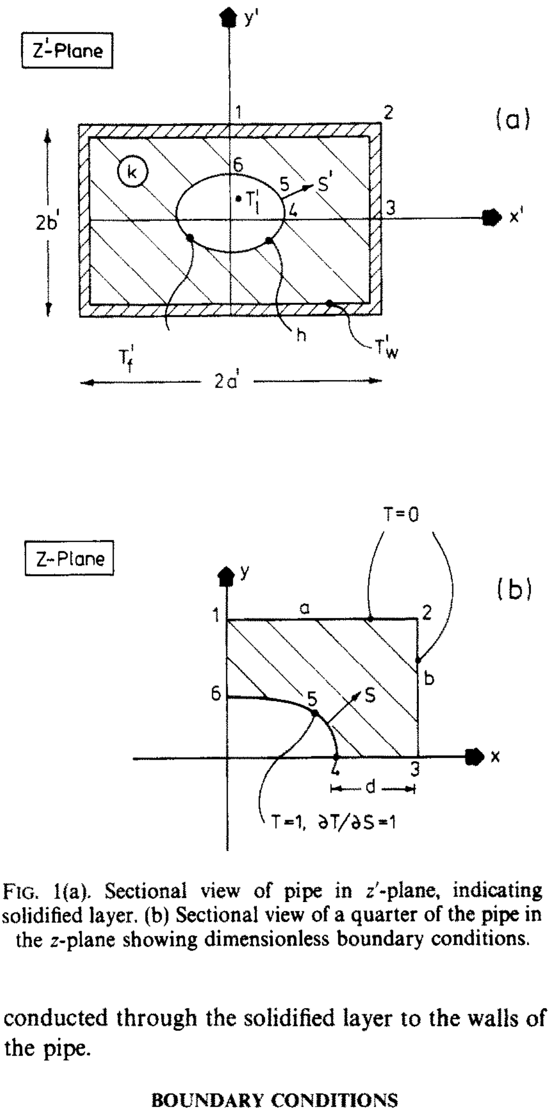

FIG. 1(a). Sectional view of pipe in z'-plane, indicating solidified layer. (b) Sectional view of a quarter of the pipe in the z-plane showing dimensionless boundary conditions.

FORMULATION OF THE PROBLEM

Conformal transformation techniques have been used in many branches of engineering not least for their simplicity and versatility. They are particularly attractive of course when applied to Laplacian fields when the governing equation remains invariant, although non-Laplacian fields have been greatly simplified in some cases. The combination of conformal transformations with the equally powerful hodograph technique, which has been mainly applied to the solution of problems in fluid mechanics [15] and more recently in electrostatic problems [16], provides a technique of great benefit to the problem considered here.

A section of long pipe is illustrated in Fig. 1(a) where the dimensions in the z'-plane are chosen to be $2a'$ and $2b'$. The walls of the pipe are cooled to a constant temperature T'_w whilst the warmer liquid flows axially down the pipe. The shape of the resulting interface changes with time until, in the steady case, the heat transferred to it by the flowing liquid equals the heat conducted through the solidified layer to the walls of the pipe.

BOUNDARY CONDITIONS

On the interface, two boundary conditions are required for a boundary of initially unspecified location. The first is taken to be T'_f = constant, which is satisfied to correspond to the condition that on the interface the temperature is equal to the fusion temperature. The second condition is concerned with the heat flux. In the real case this might be expected to vary, but not greatly, around the boundary. However, in the absence of more detailed experimental data appropriate to the fully-developed turbulent flow through a duct of any cross-sectional profile, this condition is taken as the usual engineering approximation $Nu = Q/kd\Delta T$, where the temperature difference in the present case is $\Delta T = T'_l - T'_f$ and where the 'bulk temperature T'_l of the liquid is defined as $T'_l = \int \rho u T' \mathrm{d}A/\dot{m}$ with the usual notation.

Hence, on the interface, the shape of which is unknown, the boundary conditions are

(a) $q = k\partial T'/\partial s' = h(T'_l - T'_f)$;

and

(b) $T' = T'_f$.

where s' is a coordinate in the z'-plane normal to the interface and directed as shown in Fig. 1(a). Defining

$$T = (T' - T'_w)/(T'_f - T'_w)$$

and

$$\alpha = k(T'_f - T'_w)/h(T'_l - T'_f)$$

it follows that the above conditions become

(a) $\partial T/\partial s = 1$

and (1)

(b) $T = 1$

where $s = s'/\alpha$, $x = x'/\alpha$, $y = y'/\alpha$, $a = a'/\alpha$ and $b = b'/\alpha$.

Similarly on the pipe wall the boundary condition becomes

$$T = 0. \quad (2)$$

Thus the problem in the z-plane is as shown in Fig. 1(b) where, because of symmetry, only a quarter of the pipe is shown.

THEORY

The conjugate functions chosen to represent the problem are T the temperature and ψ the flux function (ψ = constant gives locus of heat flux vector). Both T and ψ satisfy Laplace's equation and of course the usual Cauchy–Riemann equations. Hence a new complex potential W can be defined following [4], as

$$W(z) = -T + i\psi \quad (3)$$

which, like T and ψ, will be analytic. Differentiating equation (3) it follows that

$$\mathrm{d}W/\mathrm{d}z = -\partial T/\partial x + i\partial\psi/\partial x$$

and using one of the Cauchy–Riemann equations, viz.

$$\partial T/\partial x = -\partial\psi/\partial y \quad \text{and} \quad \partial T/\partial y = \partial\psi/\partial x \quad (4)$$

it follows that

$$\mathrm{d}W/\mathrm{d}z = -\partial T/\partial x + i\partial T/\partial x = \omega\,(\text{say}). \quad (5)$$

Taking the natural logarithm of both sides of equation (5) gives

$$\Omega = \log_e \mathrm{d}W/\mathrm{d}z = \log|\omega| + i\arg\omega \quad (6)$$

$$= \log|\operatorname{grad} T| + i\theta$$

i.e.

$$\Omega = \log|\partial T/\partial s| + i\theta \quad (7)$$

where $\theta = \arg\omega$, the argument of ω. The problem in the z-plane, with an unknown boundary shape (456) in Fig. 1(b) has now been re-posed in the Ω-plane,* where the coordinates are

$$\left|\frac{\partial T}{\partial s}\right| \quad \text{and} \quad \theta = \arctan\left[\frac{-\partial T/\partial y}{\partial T/\partial x}\right].$$

Thus the Ω-plane, seen in Fig. 2(a), comprises a semi-infinite region (123456) the boundary of which is rectilinear, corresponding to conditions of either constant temperature gradient or constant temperatures, respectively.

From equation (6) it can be seen that the profile in the z-plane can be computed from

$$\int \mathrm{d}z = \int \frac{\mathrm{d}W}{\mathrm{e}^{\Omega}} \quad (8)$$

only if W is a known function of Ω. To obtain this functional relation it is now necessary to consider the boundary in the W-plane, corresponding to that in the z-plane. This is shown in Fig. 2(b), where it can be seen that the boundary is composed of two rectilinear portions at constant T and two at constant ψ. The width of the rectangular region is unity, corresponding to the temperature difference, in dimensionless form, between the interface and the pipe wall.

An auxiliary transformation is now effected between both the Ω- and W-plane and a half-plane called the t-plane. The t-plane can be seen in Fig. 2(c), where, for convenience, the origin in the W-plane (point 2) is chosen to map into the point at infinity in the t-plane. The Schwarz–Christoffel transformation is used to effect both transformations, and in its application firstly between the Ω-plane and the t-plane, three of the t-values can be assigned arbitrarily (see for example [17]). In this respect the values for t at the points 2, 4 and 6 are chosen to be ∞, 1 and 0, respectively. Hence the relation between the Ω-plane and the t-plane becomes

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = \frac{C_1}{\sqrt{(t-1)}\sqrt{t}}$$

which on integration and after constants have been evaluated gives:

$$\Omega = -\log_e[\sqrt{(t-1)} + \sqrt{t}]. \quad (9)$$

On the second application of the Schwarz–Christoffel transformation to the W-plane, it is found that

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \frac{C_2}{\sqrt{(t-1)}\sqrt{t}\sqrt{(t-m)}\sqrt{(t-n)}} \quad (10)$$

where the points 1 and 3 in the Ω-plane are chosen to map into the points $t = m$ and $t = n$, respectively, the values of which becoming parameters in the problem. From the configuration of points in the t-plane, the only constraints on m, n are that $m < 0 < 1 < n$. The constant C_2 is a scaling factor which is evaluated later.

Equations (8), (9) and (10) then give, for the coordinates of points in the z-plane

$$\int \mathrm{d}z = C_2(I_1 + I_2) \quad (11)$$

* Referred to, in fluid mechanics, as a logarithmic hodograph plane.

where

$$I_1 = \int \frac{dt}{\sqrt{t}\sqrt{(t-m)}\sqrt{(t-n)}},$$

$$I_2 = \int \frac{dt}{\sqrt{(t-1)}\sqrt{(t-m)}\sqrt{(t-n)}}.$$

From a consideration of Figs. 1(b) and 2(c) it can be seen using equation (11) that

$$ib = C_2[I_1 + I_2]_{t=n}^{+\infty}$$

and (12a)

$$a = C_2[I_1 + I_2]_{t=m}^{-\infty}.$$

Writing

$$E_1 = [I_1 + I_2]_n^{+\infty} \quad \text{and} \quad E_2 = [I_1 + I_2]_m^{-\infty}$$

it can be seen that E_1 is real and E_2 is imaginary; hence

$$C_2 = ib/E_1$$

and (12b)

$$C_2 = -ia/E_3$$

where $E_3 = R_2/i$ and real.

Equations (12b) indicate firstly the value of C_2 and secondly the fact that for a square-section pipe $E_1 = -E_3$. For integration along (456) to obtain the interfacial shape, $0 \le t \le 1$ and in this range I_1 is imaginary and I_2 is real. If $t = \bar{t}$ is a point in (0, 1) then

$$\int_0^{x(\bar{t})} dx = iC_2 \int_0^{\bar{t}} \frac{dt}{\sqrt{[(-1)(t-n)(t)(t-m)]}}$$

and (13)

$$\int_0^{x(\bar{t})} dy = -iC_2 \int_1^{\bar{t}} \frac{dt}{\sqrt{[(t-n)(t-1)(t-m)]}}.$$

Non-dimensionally these equations become

$$\frac{1}{a}\int_0^{x(\bar{t})} dx = \frac{I_4}{E_3} \quad \text{and} \quad \frac{1}{b}\int_1^{y(\bar{t})} dy = \frac{I_5}{E_1} \qquad (14)$$

where I_4, I_5 are the respective integrals in equations (13).

The heat flow through the solidified layer into the cold pipe is

$$Q = -L\int_3^2 k\frac{\partial T'}{\partial x'}\, dy' - L\int_1^2 k\frac{\partial T'}{\partial y'}\, dx'$$

where L is the length of the pipe.

In dimensionless form

$$\bar{Q} = \frac{Q}{kL(T'_f - T'_w)} = -\int_3^2 \frac{\partial T}{\partial x} dy - \int_1^2 \frac{\partial T}{\partial y} dx.$$

Using equation (4)

$$\bar{Q} = \int_3^2 \frac{\partial \psi}{\partial y} dy - \int_1^2 \frac{\partial \psi}{\partial x} dx$$

$$= \psi(2) - \psi(3) - \psi(2) + \psi(1)$$

$$= \psi(1) - \psi(3)$$

$$= \psi(6) - \psi(4) \quad \text{[from Fig. 2(b)]}.$$

But

$$\psi(6) - \psi(4) = \mathrm{Im}[W(6) - W(4)] = \mathrm{Im}\int_1^0 \frac{dW}{dt} dt.$$

Hence, from equation (10)

$$\bar{Q} = -\mathrm{Im}(C_2) \times \int_0^1 \frac{dt}{\sqrt{(t-1)}\sqrt{t}\sqrt{(t-m)}\sqrt{(t-n)}}. \qquad (15)$$

Also from Fig. 2(b), $T(4) - T(3) = 1$. hence

$$-1 = Re[W(4) - W(3)] = Re\int_n^1 \frac{dW}{dt} dt.$$

Using equation (10) again, it follows that

$$-1 = \mathrm{Im}(C_2) \times \int_n^1 \frac{dt}{\sqrt{(t-1)}\sqrt{t}\sqrt{(t-m)}\sqrt{(n-t)}}. \qquad (16)$$

The integrals in equations (15) and (16) are given the symbols I_6, I_3, respectively. Eliminating $\mathrm{Im}(C_2)$ from these last two equations gives

$$\bar{Q} = I_6/I_3. \qquad (17)$$

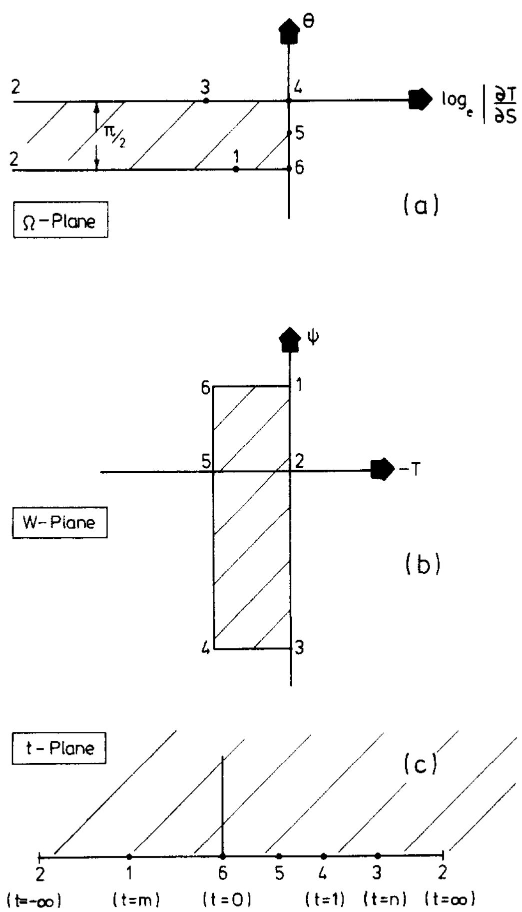

FIG. 2(a). The 'logarithmic hodograph' or Ω-plane. (b) The complex potential or W-plane. (c) The auxiliary t-plane indicating homologues of points in Ω- and W-planes.

From equations (12b) and (16) and using the fact that $a = a'/\alpha$ and $b = b'/\alpha$, it follows that

$$\frac{k(T'_f - T'_w)}{ha'(T'_l - T'_f)} = \frac{I_3}{E_3} = \frac{1}{a}$$

and

$$\frac{k(T'_f - T'_w)}{hb'(T'_l - T'_f)} = \frac{-I_3}{E_1} = \frac{1}{b}.$$

The parameters on the left-hand sides of these equations are given the symbols K_1, K_2, respectively. Hence

$$K_1 = I_3/E_3 \quad \text{and} \quad K_2 = -I_3/E_1. \tag{18}$$

Also

$$K_2/K_1 = a'/b' = a/b.$$

Thus the procedure to effect a solution to a given problem is as follows

(a) specify values for m and n (in preference to a, b when as indicated later some interpolation would be required)
(b) calculate I_3, I_6, E_1, E_3 (see also Appendix);
(c) hence calculate $\bar{Q}$ from equation (17)
(d) calculate the coordinates (x, y) of the interface using equations (14).

RESULTS AND DISCUSSION

(a) *Square-section pipe*

From equations (12a) it can be seen, in the case $a = b$, that the values of m, n must satisfy $n = 1 - m$. The coordinates of the interface, found from repeated evaluation of equations (13), are indicated in Fig. 3 for seven pairs of (m, n) and the corresponding values of K_1. Evidently (m, n) are not single-valued functions of K_1 in that for similar values of K_1, (m, n) take on totally different values. This interesting conclusion is explained further by the dependence of $\bar{Q}$ on K_1 as shown in Fig. 4, where it can be seen that $\bar{Q}$ takes two values for the same value of K_1 for a range of K_1. A plot of $\bar{Q}$ against the dimensionless solidified layer thickness d/a shows in Fig. 5, as expected, that $\bar{Q}$ decreases monotonically as the thickness increases.

From the definition of K_1 (or K_2) it can be seen that as the pipe-wall temperature (T'_w) increases, K_1 decreases. There are now two distinct cases to be considered (see Figs. 4 and 5)

(a) if the thickness of the layer is less than a critical thickness of about 0.61, corresponding to a value of $K_1 = 0.397$, then as T'_w increases, the layer thins and Q, $\bar{Q}$ increase;
(b) if the thickness of the layer is greater than this critical value, then as T'_w increases, the layer thickens and Q, $\bar{Q}$ decrease.

This 'critical thickness phenomenon' has its well-known counterpart in the external lagging of pipes.

(b) *Non-square section pipes*

To solve for the non-square pipes, values of (m, n) are chosen such that $n \neq 1 - m$. As a particular example for inclusion here, it was decided to concentrate on the pipe for which $a/b = K_2/K_1 = 2$. The pairs of values for (m, n) corresponding to this case (and all other cases) are easily determined since over small ranges it was found that for a given m (or n) the function K_2/K_1 varied linearly with $\log_e n$ (or $\log_e m$). The interfacial shapes for various values of K_1 are shown in Fig. 6, where both the x- and y-coordinates are normalized with respect to the pipe dimension a. Also the dependence of $\bar{Q}$ on K_1 can be seen in Fig. 7.

Not surprisingly, the same feature as for the square-section case occurred, namely that a critical thickness was found.

CONCLUSIONS

A method has been presented for the determination of the steady-state shape of the interface forming between a solidified layer and a warmer flowing liquid inside a cooled pipe of rectangular cross-section. The method involved the application of conformal transformation techniques in conjunction with the logarithmic hodograph plane in such a way that neither iteration nor any approximation techniques were required. For various values of the parameters K_1 and K_2, the heat transfer to the cooled wall of the pipe was calculated and it was found that, in the square-section case, a critical thickness for the layer of about 61% of the pipe width occurred. Below this critical value, as the wall temperature increases, the layer thins, whilst above the value, the layer thickens. As expected, the heat transfer increases as the layer thins and vice versa. Similar findings were also made in the non-square case.

The method, as it stands, can be further generalized to cover the cases when the interfacial heat transfer coefficient is not constant. A non-constant value would simply imply a non-rectilinear portion in the Ω-plane, easily accommodated by the method of [20]. Finally, polygonal cross-section pipes can also be analysed and this is to form the basis of a future paper.

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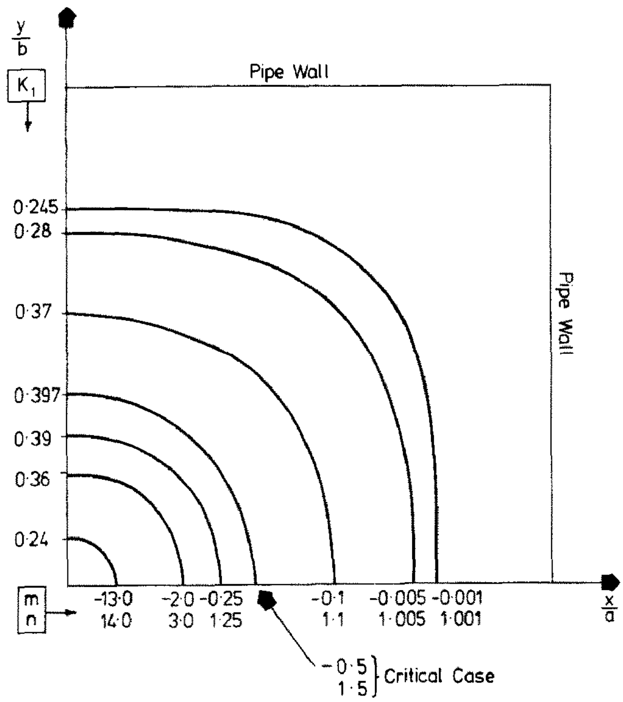

FIG. 3. The solidified layer profiles in the square pipe for various K_1 and the corresponding (m, n).

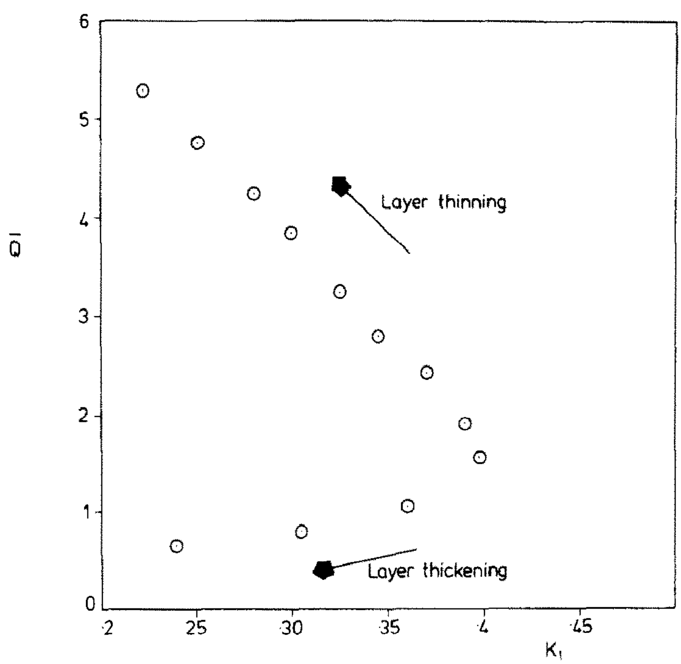

FIG. 4. The variation of the dimensionless heat transfer for $\bar{Q}$ with K_1 for the square pipe.

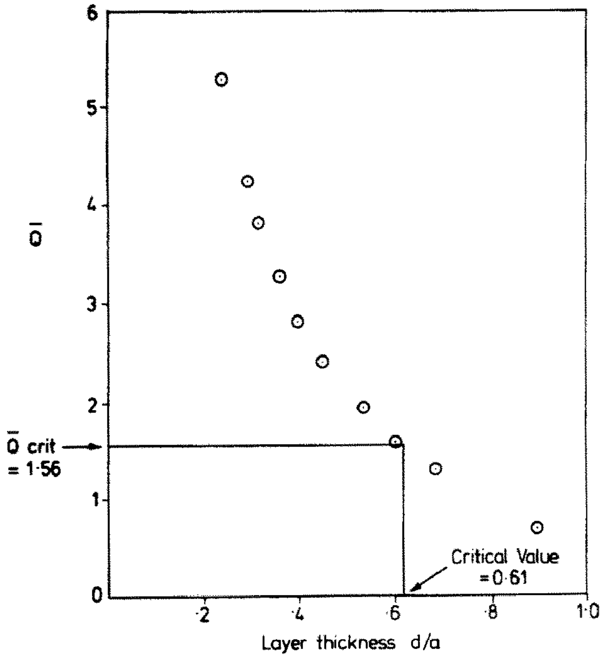

FIG. 5. The variation of $\bar{Q}$ with the dimensionless layer thickness d/a, showing critical values, for the square pipe.

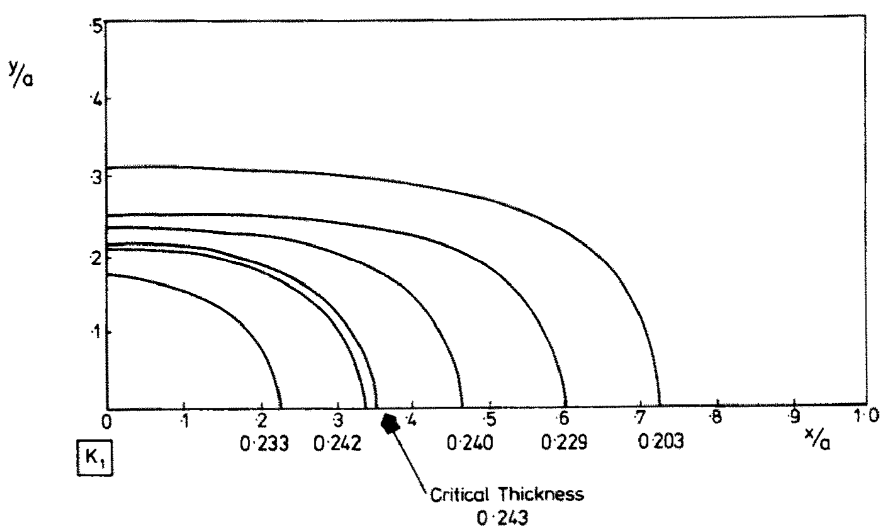

FIG. 6. The solidified layer profiles in the pipe for which $a/b = K_2/K_1 = 2$ with the corresponding K_1.

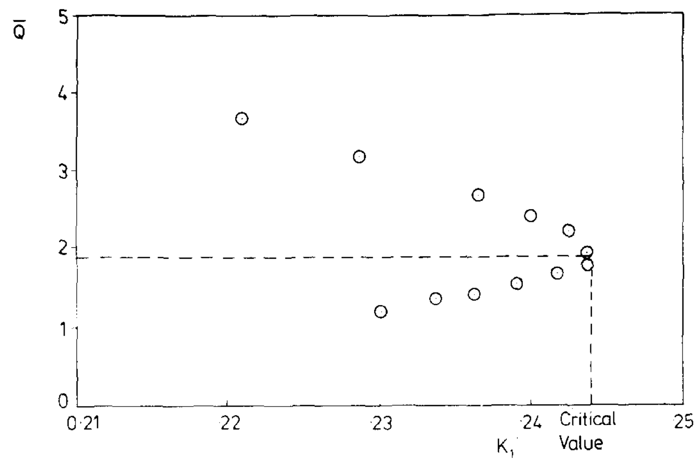

FIG. 7. The variation of $\bar{Q}$ with K_1 in the pipe for which $a/b = K_2/K_1 = 2$ showing critical values.

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APPENDIX

All of the integrals I_1, I_2, I_3, I_4, I_5, I_6 have poles in their integrands at the upper and lower limits. Thus special treatment is required for their satisfactory evaluation. The integrals I_1, I_2, I_4, I_5 involve the square root of a cubic in t in the denominator whilst in I_3 and I_6 there is a quartic. In all cases the substitution $t = (A \sin \phi + B)/(C \sin \phi + D)$ is made $(-\pi/2 \leq \phi \leq \pi/2)$ and the integrals are reduced to the Legendre integral of the first kind. Then

$$I_i = \frac{(AD - BC)}{G} F(\phi, K) \quad \text{(for all } i\text{)}$$

where A, B, C, D, G, K are constants whose values depend on the roots of the polynomials in the integrands and also on the range of integration. ϕ is the usual amplitude. (An excellent treatise on the evaluation of such integrals is given in [18].)

The evaluation of the Legendre functions was effected by a numerical technique based on an arithmetic–geometric series discussed in [19]. A batch of programs written in basic for processing on a DEC PDP8 computer were used for this evaluation.

LA SOLIDIFICATION BIDIMENSIONNELLE DANS DES CONDUITES A SECTION RECTANGULAIRE

Résumé—Une méthode de transformation conforme est appliquée à la détermination de la forme de l'interface entre une couche solidifiée à l'intérieur d'une conduité refroidie à section rectangulaire et un liquide plus chaud en écoulement. Le transfert thermique par convection du liquide sur l'interface, dans le cas stationnaire, contre-balance le transfert par conduction à travers la couche solidifiée. Ce transfert de chaleur est calculé en fonction d'un paramètre adimensionnel relatif au coefficient de convection interfaciale et la conductivité thermique de la couche. On trouve que le développement de la solidification révèle une caractéristique 'd'épaisseur critique'.

ZWEIDIMENSIONALE ERSTARRUNGSVORGÄNGE IN ROHREN MIT RECHTECKQUERSCHNITT

Zusammenfassung—Es wird eine Methode der konformen Abbildung zur Bestimmung der Form der Phasengrenzfläche zwischen einer erstarrten Schicht, welche sich an der Innenseite eines gekühlten Rohres mit Rechteckquerschnitt bildet, und einer wärmeren strömenden Flüssigkeit angewandt. Der konvektive Wärmeübergang von der Flüssigkeit an die Phasengrenzfläche ist im stationären Fall genau so groß wie der Wärmetransport durch Leitung in der erstarrten Schicht. Dieser Wärmeübergang wird für Rohre mit verschiedenen Seitenverhältnissen als Funktion eines dimensionslosen Parameters berechnet, welcher den Wärmeübergangskoeffizienten an der Phasengrenzfläche und die Wärmeleitfähigkeit der Schicht enthält. Es wurde festgestellt, daß für das Ausmaß des Erstarrens eine kritische Dicke charakteristisch ist.

ИССЛЕДОВАНИЕ ПРОЦЕССА ДВУМЕРНОГО ЗАТВЕРДЕВАНИЯ В ТРУБАХ ПРЯМОУГОЛЬНОГО СЕЧЕНИЯ

Аннотация — Для определения формы границы раздела между затвердевшим слоем на внутренней поверхности охлажденной трубы прямоугольного сечения и потоком жидкости с более высокой температурой используется метод конформного преобразования. В стационарных условиях количество тепла, переданного теплопроводностью через затвердевший слой, уравновешивается конвективным потоком тепла от жидкости к поверхности раздела. Такой тепловой режим каналов с различными значениями отношения сторон характеризуется безразмерным параметром, содержащим коэффициент межфазной конвекции и теплопроводность слоя. Найдено, что степень затвердевания характеризуется определенной «критической толщиной» слоя.